Dynamic analyses of four-bar mechanism in MechAnalyzer software

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ABSTRACT

Kinematic and dynamic analyses of mechanisms or machines form a major part in Theory of Machines course. The kinematic analysis involves study of motion of links or parts and is more intuitive to understand. On the other hand, the results of dynamic analysis which deals with forces and moments, are in the form of graph plots and require better understanding of underlying mathematics to comprehend the results. For the effective teaching of kinematics of mechanisms, the authors have developed MechAnalyzer software which has a set of common planar mechanisms that can be selected and analyzed almost instantly. In this paper, the formulation and the implementation of dynamic analysis of four-bar mechanism is presented. A recursive formulation named DeNOC (Decoupled Natural Orthogonal Complement) has been used to perform the inverse and forward dynamics, the results of which have been validated with a commercial software package.

Key words: *Planar mechanism, Simulation, Dynamics, Validation, Education*

1. INTRODUCTION

A mechanism or a machine consists of various fixed and moving parts or links connected using joints, gears, cams, etc. Based on the application, the dimensions of the links in the form of link length are determined using synthesis of mechanisms or using some standard values and the motion is studied, which form the kinematic analysis. Virtual prototypes of the mechanisms can be developed in a CAD simulation software and the effect of any change in length of the links on the operation of the mechanism can be checked until desired movement is achieved. Hence, software that can be readily used for motion simulation and analysis help in better and effective understanding of the kinematics. Though many commercial software such as ADAMS, RecurDyn, etc. are available for this purpose, their usage in teaching kinematics would require the students to have the knowledge of using those software. Instead, if some of the commonly used mechanisms can be selected, with an option to change the link parameters, the students would be able to learn the concepts more effectively and efficiently. Considering this, MechAnalyzer software was developed by the authors which aims to aid in effective teaching of mechanisms kinematics. Its comparison with other similar software are reported in Hampali et al. (2015) and Rakshith et al. (2015). Another such software is reported by Petuya et al (2014).

Once the motion study or the kinematic analysis is performed, the study of forces in the form of dynamic analysis has to be taken up. The dynamic analysis can be broadly classified as forward and inverse dynamics. The former is the study of the effect of external forces causing the motion of the mechanism, hence useful in the animation of the motion in a simulation environment. On the other hand,

the latter is the determination of the joint forces or torque required to achieve a given motion, which are useful to design the mechanism links so that they are strong enough to perform the required tasks.

The dynamic analysis is performed by deriving the governing equations of motion for the mechanism or the system. Some of the well-known methodologies are Newton-Euler (NE), Euler-Lagrange (EL), Kane's method, etc. The NE method is based on the vector approach and consists of three translatory equations (Newton's equations) and three rotational motion equations (Euler's equations) for each of the link. Further constraint equations are derived for the joints and loop closure. Together, they form a set of Differential Algebraic Equations (DAE). On the contrary, EL method uses energy based approach and the equations of motion of a mechanism can be derived using independent coordinates whose number is equal to the degree-of-freedom (DOF) of the mechanism. The EL methodology results into a set of Ordinary Differential Equations (ODE). Kane's method also provides ODE that have to solved.

To be implemented in a computer software, the solution methods for ODE are better than those available for solving DAE. Hence, several researchers have proposed formulation of dynamics by beginning with NE approach and later on reducing the number of equations using some constraints or other technique. By doing so, the final set of equations are in the form of ODE and hence can be implemented more easily. A thorough overview on such methods is reported in Chaudhary and Saha (2008). One such method is Decoupled Natural Orthogonal Complement (DeNOC) approach reported by Saha (1999), which was originally proposed for serial chain manipulators or systems.

DeNOC method uses NE equations as a set of block matrices equations for the uncoupled system, where each link is considered to have six DOF and hence free to move around. The constraints are then applied in the form of linear and angular velocities of rigid bodies, and the associated joint rates, after which the minimal set of equations are obtained. There are several advantages of DeNOC over conventional NE or EL formulation, such as: 1) It provides an $O(n)$ algorithm for both inverse and forward dynamics, where n is the number of links in a serial system. 2) All the scalar elements in the matrices and vectors associated with the equations of motion have an analytical expression, which helps in understanding some of the underlying concepts such as articulated body inertia, etc. Also, these help in debugging a computer program. 3) As the methodology is based upon mechanics and linear algebra concepts, it can be understood by under-graduate students. The DeNOC methodology was later extended to closed-loop mechanisms (Chaudhary and Saha, 2008) and tree-type systems (Shah at el. 2012).

In this paper, the formulation for the dynamic analysis of a four-bar mechanism is derived using DeNOC method and discussed in Section 2. The implementation and the usage of software is reported in Section 3 and Section 4 has the validation of results, followed by the conclusions.

2. DYNAMICS OF FOUR-BAR MECHANISM

Four-bar mechanism is one of the important mechanisms that is used to teach the concepts related to kinematics and dynamics. Hence, it was chosen as the first mechanism for the implementation. The inverse dynamics formulation is discussed first, followed by the forward dynamics.

2.1 Inverse Dynamics of Four-bar mechanism

Four-bar mechanism which is a closed loop system, is first divided into an equivalent open architecture by introducing cuts at appropriate joints. Here, the two subsystems are obtained by introducing cut-open arrangement at joint J2 as illustrated in Figure 1. The two subsystems obtained are serial in nature with some constraint existing between them. The cut joints are replaced by unknown constraint forces, also called Lagrange multipliers. Such subsystem allows one to use already available well established algorithms for the serial systems. The two subsystem thus obtained are a one link subsystem, referred as Subsystem I, and a two link subsystem, referred as Subsystem II. The Lagrange multiplier between these subsystems is represented using λ. The links are assumed to have their center of masses at a distance of r from one end of the respective links,

such as r_1 for #2 from J1, r_3 for #4 from J4, and r_2 for #3 from J3. The masses of links #2, #3 and #4 are represented by m_2, m_3, and m_4, respectively. To derive the equations of motion for the closed loop system, two subsystems and have been considered, i.e. one body in Subsystem I and two bodies in Subsystem II and the subsystems are represented using superscripts of I and II, respectively. The steps followed in the inverse dynamics analysis are explained next.
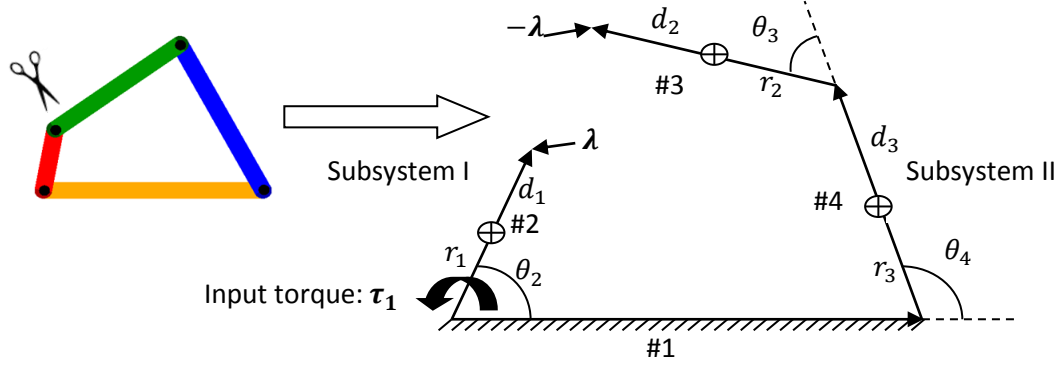

Figure 1: *Four-bar mechanism converted into two serial subsystems*

a. *Uncoupled Newton-Euler Equations of motion*
   Referring to Figure 2, consider i^{th} link with m_i as mass of the link whose center of mass (C_i) is located by vector $\mathbf{c}_i$ in a fixed frame of reference. External force $\mathbf{f}_i$ and moment $\mathbf{n}_i$ are acting on the link causing its motion.
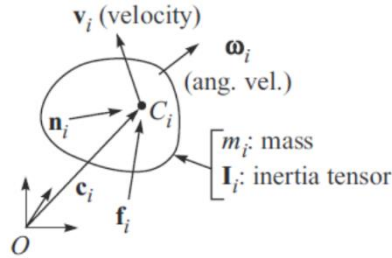

Figure 2: External force and moment acting on a rigid body or link

The NE equations of motion for the ith link can be derived using free body diagram as

$$\mathbf{f}_i = m_i \ddot{\mathbf{c}}_i \tag{1}$$

$$\mathbf{n}_i = \mathbf{I}_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times \mathbf{I}_i \boldsymbol{\omega}_i \tag{2}$$

Equations (1) and (2) can be written in a compact form using matrices and vectors as

$$\mathbf{M}_i \dot{\mathbf{t}}_i + \mathbf{W}_i \mathbf{M}_i \mathbf{t}_i = \mathbf{w}_i \tag{3}$$

where $\mathbf{M}_i$ and $\mathbf{W}_i$ are 6×6 mass inertia matrix, and the matrix of angular velocities respectively and are given by

$$\mathbf{M}_i \equiv \begin{bmatrix} \mathbf{I}_i & \mathbf{0} \\ \mathbf{0} & m_i \mathbf{1} \end{bmatrix}; \text{ and } \mathbf{W}_i \equiv \begin{bmatrix} \boldsymbol{\omega}_i \times \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{4}$$

where $\mathbf{I}_i$ denotes the 3×3 inertia tensor of the body and $\boldsymbol{\omega}_i \times \mathbf{1}$ is 3×3 cross product tensor associated with angular velocity. Also, $\mathbf{1}$ and $\mathbf{0}$ are the 3×3 identity and zero matrices respectively. Furthermore, the 6-dimensional twist vector $\mathbf{t}_i$, and wrench vector $\mathbf{w}_i$ are defined as

$$\mathbf{t}_i \equiv \begin{bmatrix} \boldsymbol{\omega}_i \\ \dot{\mathbf{c}}_i \end{bmatrix} \text{ and } \mathbf{w}_i \equiv \begin{bmatrix} \mathbf{n}_i \\ \mathbf{f}_i \end{bmatrix} \tag{5}$$

The NE equations of the uncoupled system with n moving bodies can be combined into a single block matrix equation as

$$\mathbf{M\dot{t}} + \mathbf{WMt} = \mathbf{w} \tag{6}$$

where $\mathbf{M}$ and $\mathbf{W}$ are $6n \times 6n$ generalized mass inertia matrix, and the generalized matrix of angular velocities, respectively.

For the four-bar mechanism in Figure 1, the NE equations for the uncoupled subsystems I and II are written as

$$\mathbf{M}^I \equiv \mathbf{M}_1 \quad \text{and} \quad \mathbf{W}^I \equiv \mathbf{W}_1$$

$$\mathbf{t}^I \equiv \mathbf{t}_1 \text{ and } \mathbf{w}^I \equiv \mathbf{w}_1 \tag{7}$$

$$\mathbf{M}^{II} \equiv diag.\,[\mathbf{M}_4, \mathbf{M}_3] \quad \text{and} \quad \mathbf{W}^{II} \equiv diag.\,[\mathbf{W}_4, \mathbf{W}_3]$$

$$\mathbf{t}^{II} \equiv [\mathbf{t}_4^T, \mathbf{t}_3^T]^T \quad \text{and} \quad \mathbf{w}^{II} \equiv [\mathbf{w}_4^T, \mathbf{w}_3^T]^T \tag{8}$$

b. *Kinematic constraints:*
For a serial system represented using Equation (6), kinematic constraints in the velocity level are applied to get a system of ODE in the minimal form. Velocity level constraints are obtained as a function of the joint rates as

$$\mathbf{t} = \mathbf{N\dot{\theta}} \tag{9}$$

where N is a $6n \times n$ matrix, known as Natural Orthogonal Complement (Angeles and Ma, 1988), whose utility is explained in the next sub-section. In the DeNOC method, N is further decoupled into two matrices as

$$\mathbf{N} \equiv \mathbf{N_l N_d} \tag{10}$$

where $\mathbf{N}_l$ is a $6n \times n$ lower diagonal matrix and $\mathbf{N}_d$ is an n dimensional diagonal matrix. These matrices allow one to develop recursive inverse and forward dynamics algorithms that are required in control and simulation of various robotic systems.

For the four-bar mechanism considered here, the links in Subsystem II are coupled by revolute joints. The angular and linear velocity of #3 can be derived from link #4. The kinematic constraint can be written in following form:

$$\boldsymbol{\omega}_3 = \boldsymbol{\omega}_4 + \mathbf{e}_3 \dot{\theta}_3 \tag{11}$$
$$\dot{\mathbf{c}}_3 = \dot{\mathbf{c}}_4 + \boldsymbol{\omega}_4 \times \mathbf{r}_3 + \boldsymbol{\omega}_3 \times \mathbf{d}_2 \tag{12}$$

where $\mathbf{e}_3 \equiv [0 \quad 0 \quad 1]^T$,i.e. the axis of the revolute joint which is along the Z axis. The compact form of the twist vector of Subsystem II can be written as

$$\mathbf{t}^{II} = \mathbf{N}^{II}\dot{\boldsymbol{\theta}}^{II} \quad \text{where } \mathbf{N}^{II} \equiv \mathbf{N}_l^{II}\mathbf{N}_d^{II} \tag{13}$$

The DeNOC matrices for Subsystem II is obtained as below

$$\mathbf{N}_l^{II} \equiv \begin{bmatrix} \mathbf{1} & 0 \\ \mathbf{B}_{21} & \mathbf{1} \end{bmatrix} ; \mathbf{N}_d^{II} \equiv \begin{bmatrix} \mathbf{p}_1^{II} & 0 \\ 0 & \mathbf{p}_2^{II} \end{bmatrix} ; \text{and } \dot{\boldsymbol{\theta}}^{II} \equiv \begin{bmatrix} \dot{\theta}_4 \\ \dot{\theta}_3 \end{bmatrix}$$

$$\text{where } \mathbf{B}_{21} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ -(\mathbf{r}_2 + \mathbf{d}_3) \times \mathbf{1} & \mathbf{1} \end{bmatrix} \text{ and } \mathbf{p}_i \equiv [\mathbf{e}_i \quad \mathbf{e}_i \times \mathbf{d}_i]^T$$

(14)

Similarly for Subsystem I, which just has a single link with a revolute joint, the constraints equations are

$$\mathbf{N}_l^I \equiv [\mathbf{1}] ; \mathbf{N}_d^I \equiv [\mathbf{p}_1^I] ; \text{and } \dot{\boldsymbol{\theta}}^I \equiv [\dot{\theta}_2]$$

(15)

c. Coupled Equations of Motion

The uncoupled NE equations of motion of Equation (9) are pre-multiplied by $\mathbf{N}^T$ on both sides. The constraints equations are orthogonal to the twist vector t and hence are eliminated from the overall equations. Hence uncoupled equations (NE) are converted into minimal set of equations.

$$\mathbf{N}^T(\mathbf{M}\dot{\mathbf{t}} + \mathbf{W}\mathbf{M}\mathbf{t}) = \mathbf{N}^T\mathbf{w}$$

(16)

By using $\dot{\mathbf{t}} = \mathbf{N}\ddot{\boldsymbol{\theta}} + \dot{\mathbf{N}}\dot{\boldsymbol{\theta}},$ Equation (16) can be written in block matrices and vectors form as

$$\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} = \boldsymbol{\tau} + \boldsymbol{\tau}_g$$

(17)

where $\boldsymbol{\tau}_g$ is the gravity vector and $\boldsymbol{\tau}$ is the torques at the joints and the other terms are

$$\mathbf{I} \equiv \mathbf{N}^T\mathbf{M}\mathbf{N} ; \mathbf{C} \equiv \mathbf{N}^T(\mathbf{M}\dot{\mathbf{N}} + \mathbf{W}\mathbf{M}\mathbf{N}) ; \text{and } \boldsymbol{\tau} = \mathbf{N}^T\mathbf{w}$$

(18)

For Subsystem II, the input torques are zero, i.e., the torques at the joints J4 and J3 are zero. The unknown constraint forces or Lagrange multiplier $\boldsymbol{\lambda}$, a 3×1 vector is calculated using

$$\mathbf{I}^{II}\ddot{\boldsymbol{\theta}}^{II} + \mathbf{C}^{II}\dot{\boldsymbol{\theta}}^{II} = \boldsymbol{\tau}_g + (\mathbf{J}^{II})^T(-\boldsymbol{\lambda})$$

(19)

Vector $\boldsymbol{\lambda}$ from Equation (19) is substituted in Subsystem I to get the input torque as

$$I^I\ddot{\theta}^I + C^I\dot{\theta}^I = \tau_1 + \tau_g + (\mathbf{J}^I)^T(\boldsymbol{\lambda})$$

(20)

where $\mathbf{J}^I = \begin{bmatrix} -l_2\sin(\theta_2) \\ l_2\cos(\theta_2) \end{bmatrix}$ and $\mathbf{J}^{II} = \begin{bmatrix} -l_4\sin(\theta_4) - l_3\sin(\theta_3 + \theta_4) & l_3\sin(\theta_3 + \theta_4) \\ l_4\cos(\theta_4) + l_3\cos(\theta_3 + \theta_4) & l_3\cos(\theta_3 + \theta_4) \end{bmatrix}$ are Jacobian matrices for subsystem I and II, respectively and l_2, l_3 and l_4 are lengths of links #2, #3 and #4 respectively. These Jacobian matrices relate the linear velocity of the links with the angular velocities of the joints. Using Equation (20), the input torque at joint J1 (τ_1) is determined, thus completing the inverse dynamic analysis.

2.2 Forward Dynamics of Four-bar mechanism

The forward dynamics is done by knowing the kinematic and inertial parameters and joint torques and forces to find the trajectories of the joints of a mechanism. For known input and external forces/torques, vector $\boldsymbol{\theta}$. The vector $\boldsymbol{\theta}$ for a given torque $\boldsymbol{\tau}$ is found out by calculating $\ddot{\boldsymbol{\theta}}$ and then integrating to find velocity and position.

The two subsystems obtained above for the four-bar mechanism are rewritten to obtain the position and velocity by calculating accelerations. The constrained equations of motion obtained in Equations (19) and (20) are combined with velocity constraints Equation (21) to obtain differential algebraic equations(DAE) as

$$[\mathbf{J}^I \quad -\mathbf{J}^{II}] \begin{bmatrix} \dot{\boldsymbol{\theta}}^I \\ \dot{\boldsymbol{\theta}}^{II} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} I^I & \mathbf{0} & (\mathbf{J}^I)^T \\ 0 & \mathbf{I}^{II} & -(\mathbf{J}^{II})^T \\ \mathbf{J}^I & -\mathbf{J}^{II} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\theta}}^I \\ \ddot{\boldsymbol{\theta}}^{II} \\ \boldsymbol{\lambda} \end{bmatrix} \equiv \begin{bmatrix} \tau_1 + \tau_g - C^I \dot{\theta}^I \\ \boldsymbol{\tau}_g - \mathbf{C}^{II} \dot{\boldsymbol{\theta}}^{II} \\ -\mathbf{\dot{j}}^I \dot{\theta}^I + \mathbf{\dot{j}}^{II} \dot{\boldsymbol{\theta}}^{II} \end{bmatrix} \tag{22}$$

Let **A** denote the matrix on LHS, **x** be the vector containing accelerations terms and Lagrange multiplier, and **B** be the matrix on the RHS. Hence **x** can be found out by following equation.

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B} \tag{23}$$

Here, since matrix **A** is symmetric positive definite, it is always invertible. As the DAE obtained are highly non-linear and cannot be solved analytically, a suitable numerical technique has to be used to determine the joint accelerations, thus completing the forward dynamics analysis.

The numerical technique used for forward dynamic analysis is Runge-Kutta method. The initial conditions considered for the analyses is taken to be at rest, making initial velocities and accelerations of all the links as zero. The initial position for link #2 is at rest while #3 and #4 is found by position analysis through vector closed loop equations. These initial conditions are used to calculate the accelerations and then integrated using Runge-Kutta method to find the velocity and position consecutively, which are then used for finding the new acceleration. The iterative procedure is continued till the given number of steps and time as input by the user.

3. IMPLEMENTATION IN MECHANALYZER

The kinematic analysis of a four-bar mechanism was already existing in MechAnalyzer Version 4 (Lokesh et al, 2015). Version 5 has dynamics of a four-bar mechanism implemented, in addition to some other additional features and modules. User can easily change the length of the links and the 3D model of the mechanism gets updated instantly. The mass and inertia properties of the links can be changed to perform the dynamic analysis. The software has been developed using Visual C# programming language and the formulations of inverse and forward dynamics discussed in the previous section are implemented. A screenshot of MechAnalyzer software is shown in Figure 3.
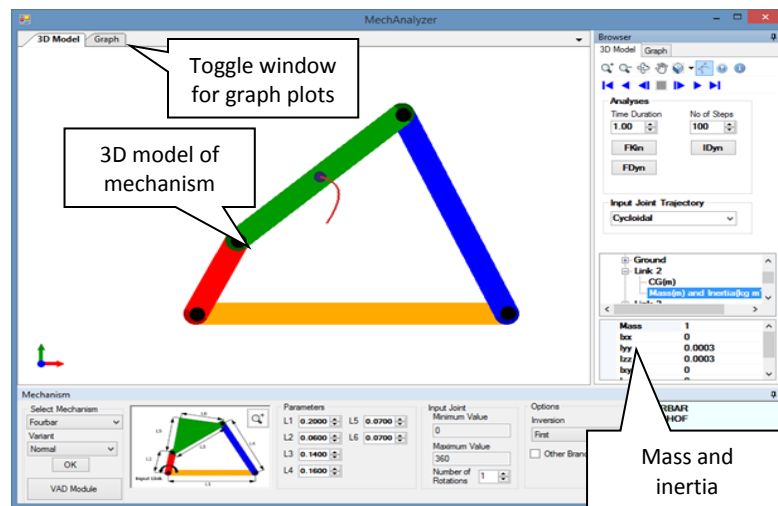

Figure 3: Graphical User Interface (GUI) of MechAnalyzer showing a four-bar mechanism

For the set values of mass and inertia properties, and the input joint trajectory, the driving torque for joint J1 is determined as a part of inverse dynamics. The results obtained can be easily viewed as graph plots in the software. The graph plots are developed using ZedGraph, an open source library for plots. For the mechanism shown in Figure 3, the input joint torque determined with and without gravity, are shown in Figures 4(a) and 4(b), respectively. Similarly, for the forward dynamics, the motion of the mechanism due to the action of gravity can be seen. The plots of the joint angle, velocity and acceleration can also be plotted.
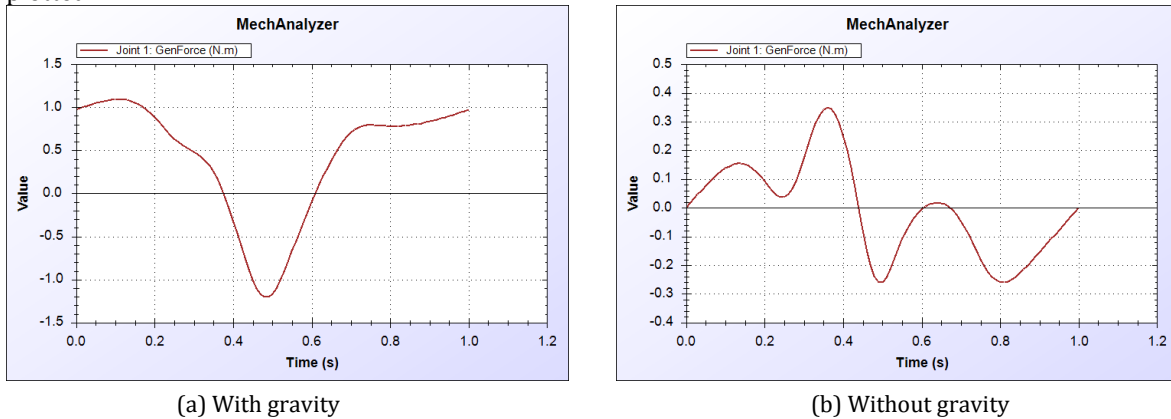

(a) With gravity   (b) Without gravity

Figure 4: Plots of inverse dynamic analysis of four-bar mechanism

Though plots of dynamic analysis are useful in design and simulation of mechanisms, they are not as intuitive as kinematic analysis results for students to understand and comprehend. However, using MechAnalyzer software, the effect of change of mass and inertia properties of links on the torque can be illustrated by comparing different plots. For example, for different values of mass of coupler link (link #3), the driving torque obtained through inverse dynamics is plotted as shown in Figure 5. It can be noted that for higher mass, more torque is required to rotate the crank link, hence making students understand an underlying concept.
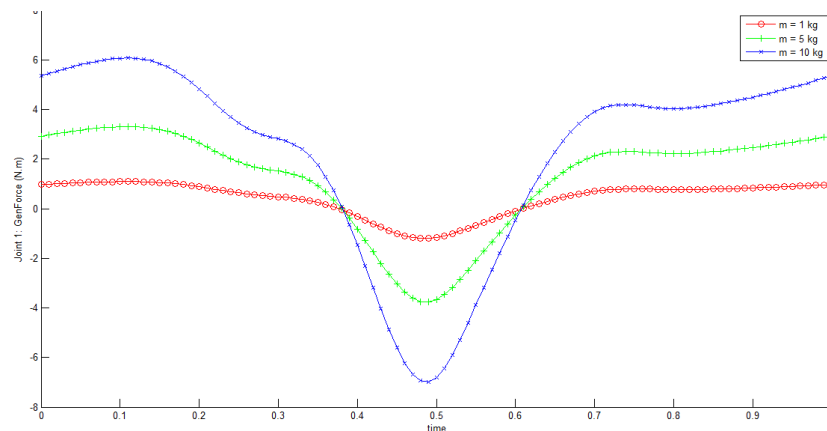

Figure 5: Comparison of input torque for different values of mass of coupler link in a four-bar mechanism

4.   VALIDATION OF RESULTS

The formulation for the dynamic analysis have been implemented using numerical techniques. Especially, the forward dynamic algorithm requires solution of DAE using a method similar to the ODE45 function of MATLAB. The results obtained from MechAnalyzer were compared with those obtained using Dynamic Simulation module in Autodesk Inventor CAD software. Comparison of inverse and forward dynamics are

shown in Figures 6(a) and 6(b), respectively. The plots matched exactly, thus validating the formulation and implementation of the dynamics algorithm in MechAnalyzer software.
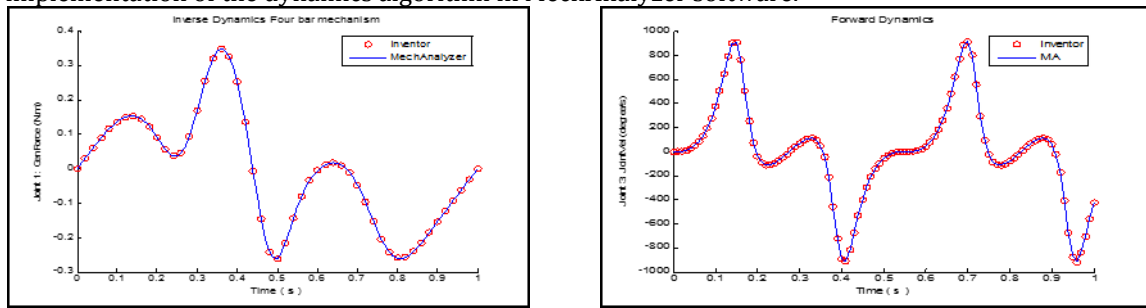
(a) Inverse dynamics without gravity

(b) Angular velocity of joint 3 for forward dynamics

Figure 6: Validation of results with Dynamic Simulation module of Autodesk Inventor software

5. CONCLUSION

The formulation and the implementation details of the dynamic analysis of four-bar mechanism in MechAnalyzer software have been presented in this paper. The formulation used is based on DeNOC methodology which is a recursive algorithm. The output of the analyses can be viewed as animation and also in the form of graph plots, which would help in effective teaching and learning of the concepts related to dynamics. As four-bar mechanism is commonly used in teaching, it has been implemented first. In the future, dynamic analyses of more such mechanisms will be implemented in MechAnalyzer. The reported implementation is available in MechAnalyzer Version 5, which can be downloaded for free from http://www.roboanalyzer.com/mechanalyzer.html.

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